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Journal of Sound and Vibration 306 (2007) 284–296
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A numerical method for quadratic eigenvalue problems of gyroscopic systems

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Received 28 December 2006; received in revised form 2 May 2007; accepted 5 May 2007
Available online 22 June 2007

Abstract

We consider the quadratic eigenvalues problem (QEP) of gyroscopic systems $(\lambda^2\mathbf{M}+\lambda\mathbf{G}+\mathbf{K})\mathbf{x}=\mathbf{0}$, where $\mathbf{M}=\mathbf{M}^\top, \mathbf{G}=-\mathbf{G}^\top$ and $\mathbf{K}=\mathbf{K}^\top\in\mathbb{R}^{n\times n}$ with $\mathbf{M}$ being positive definite. Guo [Numerical solution of a quadratic eigenvalue problem, *Linear Algebra and its Applications* 385 (2004) 391–406] showed that all eigenvalues of the QEP can be found by solving the maximal solution of a nonlinear matrix equation $\mathbf{Z}+\mathbf{A}^\top\mathbf{Z}^{-1}\mathbf{A}=\mathbf{Q}$ with quadratic convergence when the QEP has no eigenvalues on the imaginary axis. The convergence becomes linear or more slower (Guo, 2004) when the QEP allows purely imaginary eigenvalues having even partial multiplicities. In this paper, we consider the general case when the QEP has eigenvalues on the imaginary axis. We propose an eigenvalue shifting technique to transform the original gyroscopic system to a new gyroscopic system, which shifts purely imaginary eigenvalues to eigenvalues with nonzero real parts, while keeps other eigenpairs unchanged. This transformation ensures that the new method for computing the maximal solution of the nonlinear matrix equation converges quadratically. Numerical examples illustrate the efficiency of our method.

1. Introduction

The quadratic eigenvalue problem (QEP) is to find scalars $\lambda\in\mathbb{C}$ and nonzero vectors $\mathbf{x}\in\mathbb{C}^n$ satisfying

$$Q(\lambda)\mathbf{x}=(\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K})\mathbf{x}=\mathbf{0}, \tag{1.1}$$

where $\mathbf{M},\mathbf{C},\mathbf{K}$ are $n\times n$ matrices. The scalar λ and the nonzero vector $\mathbf{x}$ are called, respectively, the eigenvalue and the (right) eigenvector corresponding to λ. The nonzero vector $\mathbf{y}\in\mathbb{C}^n$ satisfying

$$\mathbf{y}^*Q(\lambda)=\mathbf{y}^*(\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K})=\mathbf{0}$$

is called the left eigenvector corresponding to λ. Here $\mathbf{y}^*\equiv\bar{\mathbf{y}}^\top$ denotes the conjugate transpose of $\mathbf{y}$. The quadratic matrix polynomial $Q(\lambda)$ in Eq. (1.1) is, generally, known as a quadratic pencil. The QEP arises in a wide variety of applications, such as dynamics analysis of mechanical systems, fluid mechanics and

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[1]This author's research was partly supported by Project NSFC 10571007.

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doi:10.1016/j.jsv.2007.05.009

vibro-acoustics, which has received much attention in recent years. A good survey of applications, spectral theory, perturbation analysis and numerical approaches can be found in Ref. [2] and the references therein.

In this paper, we develop a numerical method for a special QEP of the form

$$G(\lambda)\mathbf{x} = (\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K})\mathbf{x} = \mathbf{0}, \tag{1.2}$$

where $\mathbf{M} = \mathbf{M}^\top, \mathbf{G} = -\mathbf{G}^\top, \mathbf{K} = \mathbf{K}^\top \in \mathbb{R}^{n\times n}$ with $\mathbf{M} > \mathbf{0}$ being positive definite. There is a second-order differential equation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{G}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t)$$

associated with Eq. (1.2), which is known as a gyroscopic system. Gyroscopic systems correspond to spinning structures where the Coriolis inertia forces are taken into account, and they are widely known to exhibit instabilities [3,4].

It is well known that the $2n$ eigenvalues of $G(\lambda)$ has Hamiltonian properties, i.e., they are symmetrically located with respect to the real and imaginary axes. Furthermore, if $\mathbf{x}$ is the right eigenvector corresponding to an eigenvalue λ, then $\bar{\mathbf{x}}$ is the eigenvector (or the left eigenvector) corresponding to $\bar{\lambda}$ (or $-\lambda$). When $\mathbf{K}$ is positive definite, all eigenvalues of $G(\lambda)$ are purely imaginary and semisimple, and the system is stable. Strong stability, which refers to a system and all its neighbors being stable, has been investigated in Ref. [4]. When $\mathbf{K}$ is negative definite, the eigenvalues of $G(\lambda)$ are not necessarily purely imaginary and semisimple, hence the system is not guaranteed to be stable [2].

In this paper, we consider the QEP (1.2) with $\mathbf{K}$ being negative semidefinite, and we are interested in finding all $2n$ eigenvalues and the associated eigenvectors. A standard approach for finding all eigenpairs of the QEP is to work with a $2n \times 2n$ linearized form and compute its generalized Schur form. However, such approaches cannot generally keep the Hamiltonian structure. To this end, we can transform the QEP (1.2) to a Hamiltonian eigenvalue problem [2], and then apply the structure-preserving methods [5,6] for computing all eigenvalues. The skew-Hamiltonian structure is preserved in both methods by the use of symplectic orthogonal transformations. Note that these methods work on matrices of dimension $2n$ and produce no eigenvectors. Mehrmann and Watkins [7] proposed a structure-preserving shift-and-invert Krylov subspace method for the computation of a few eigenvalues and eigenvectors of large, sparse skew-Hamiltonian/Hamiltonian pencils, and applied the method to the QEP (1.2). Recently, Guo [1] shows that all the eigenpairs of the QEP (1.2) can be solved by finding a proper solvent $\mathbf{S}$ of the quadratic matrix equation $\mathbf{M}\mathbf{S}^2 + \mathbf{G}\mathbf{S} + \mathbf{K} = \mathbf{0}$, which is equivalent to find the maximal solution $\mathbf{Z}_+$ of a nonlinear matrix equation (NME) $\mathbf{Z} + \mathbf{A}^\top\mathbf{Z}^{-1}\mathbf{A} = \mathbf{Q}$, with $\mathbf{A} = \mathbf{M} + \mathbf{K} + \mathbf{G}$ and $\mathbf{Q} = 2(\mathbf{M} - \mathbf{K})$. The numerical approach for the NME developed in Ref. [1] is based on the cyclic reduction method [8]. The method is quadratically convergent if $\psi(\lambda) \equiv \lambda^2\mathbf{A}^\top + \lambda\mathbf{Q} + \mathbf{A}$ has no unimodular eigenvalues, and is linearly convergent if all unimodular eigenvalues of $\mathbf{Z}_+^{-1}\mathbf{A}$ are semisimple. Another efficient method for solving the NME based on the SDA algorithm [9] is shown to be linearly convergent if the unimodular eigenvalues of $\mathbf{Z}_+^{-1}\mathbf{A}$ have half of the partial multiplicity of the associated unimodular eigenvalue of $\psi(\lambda)$.

Linear convergence is poor. The main purpose of this paper is to provide a new technique to shift the purely imaginary eigenvalues, while keeping the other eigenpairs unchanged. We then apply the structure-preserving methods in Ref. [1] or [9] to find the maximal solution of the resulted NME with quadratic convergence.

The paper is organized as follows. In Section 2, we review some results in Ref. [1] on how to solve the QEP by the solvent approach, and the SDA algorithm and the related convergence analysis in Refs. [10,9]. The main technique of shifting the purely imaginary eigenvalues of the QEP is developed in Section 3. Some numerical examples are shown in Section 4 to illustrate the efficiency of our approach. Some conclusions are given in Section 5.

2. Solving the QEP

As described in Section 1, the classical approach for finding all $2n$ eigenpairs of the QEP is to use linearizations and solve the resulting $2n \times 2n$ standard or generalized eigenvalue problem. These methods operate in dimensions twice that of the original problem. Another approach is to factorize $G(\lambda)$ in Eq. (1.2). It

is well known that $G(\lambda)$ has the factorization

$$G(\lambda) = (\lambda\mathbf{M} + \mathbf{MS} + \mathbf{G})(\lambda\mathbf{I} - \mathbf{S}) \tag{2.1}$$

if and only if $\mathbf{S}$ is a solution of the quadratic matrix equation

$$\mathbf{MS}^2 + \mathbf{GS} + \mathbf{K} = \mathbf{0}. \tag{2.2}$$

Such an $\mathbf{S}$ is called a solvent of Eq. (2.2) [3]. If Eq. (2.2) has a solvent $\mathbf{S}$, then the eigenvalues of $G(\lambda)$ are those of $\mathbf{S}$ and those of the matrix pencil $\lambda\mathbf{M} + \mathbf{MS} + \mathbf{G}$. However, Eq. (2.2) may not have any solvent. Even if the solvent exists, the computation involved may be difficult. Fortunately, for the special QEP (1.2), Guo ([1], Lemma 1) proved that if the QEP (1.2) has no eigenvalues on the imaginary axis, then the matrix equation (2.2) has a real solvent whose eigenvalues are on the right half-plane. Therefore, if we can find such $\mathbf{S}$, the remaining n eigenvalues of $G(\lambda)$ are obtained by symmetry without any computation. Moreover, since $\mathbf{S}$ is real, the complex eigenvalues of $\mathbf{S}$ must appear in complex conjugate pairs. Together with the eigenvalues on the left half-plane obtained by symmetry, the Hamiltonian structure for the eigenvalues of the QEP (1.2) is preserved. The eigenvectors can be obtained from the factorization (2.1) and the properties of eigenvectors of the QEP (1.2). If $\mathbf{x}_i$ and $\mathbf{y}_i$ are, respectively, the right and left eigenvectors corresponding to an eigenvalue λ_i of the solvent $\mathbf{S}$, or

$$\mathbf{S}\mathbf{x}_i = \lambda_i\mathbf{x}_i, \quad \mathbf{y}_i^*\mathbf{S} = \lambda_i\mathbf{y}_i^*,$$

then $\mathbf{x}_i$ and $(\lambda_i\mathbf{M} + \mathbf{MS} + \mathbf{G})^{-\top}\bar{\mathbf{y}}_i$ are eigenvectors corresponding to $\pm\lambda_i$, both eigenvalues of the QEP (1.2).

It seems difficult to find the solvent of Eq. (2.2) directly whose eigenvalues are on the right half-plane. Instead, the Cayley transformation $\mathbf{S} = (\mathbf{I} + \mathbf{Y})(\mathbf{I} - \mathbf{Y})^{-1}$ is used (see Ref. [1]). Eq. (2.2) then becomes

$$\mathbf{A}^\top\mathbf{Y}^2 + \mathbf{QY} + \mathbf{A} = \mathbf{0}, \tag{2.3}$$

where

$$\mathbf{A} = \mathbf{M} + \mathbf{K} + \mathbf{G}, \quad \mathbf{Q} = 2(\mathbf{M} - \mathbf{K}) > \mathbf{0}. \tag{2.4}$$

Recall that the eigenvalues of $\mathbf{S}$ lie on the right half-plane. With the Cayley transformation, we are now interested in the solution $\mathbf{Y}$ of Eq. (2.3) whose eigenvalues are inside the unit circle. With $\mathbf{Y} = -\mathbf{Z}^{-1}A$ in Eq. (2.3), we arrive at the NME:

$$\mathbf{Z} + \mathbf{A}^\top\mathbf{Z}^{-1}\mathbf{A} = \mathbf{Q}. \tag{2.5}$$

The solution $\mathbf{Z}$ of Eq. (2.5) satisfies $\rho(\mathbf{Z}^{-1}\mathbf{A}) < 1$ or $\rho(\mathbf{Z}^{-1}\mathbf{A}) \leqslant 1$ under appropriate assumptions. Here $\rho(\cdot)$ denotes the spectral radius of a matrix.

Eq. (2.5) has been well studied in Refs. [8,10–12]. We are interested in finding the maximal symmetric positive definite solution of Eq. (2.5). A symmetric positive definite solution $\mathbf{Z}_+$ is called a maximal solution if $\mathbf{Z}_+ \geqslant \mathbf{Z}$ for any symmetric positive definite solution $\mathbf{Z}$ of Eq. (2.5). Here $\mathbf{Z}_1 \geqslant \mathbf{Z}_2$ ($\mathbf{Z}_1 > \mathbf{Z}_2$) means that $\mathbf{Z}_1 - \mathbf{Z}_2 \geqslant \mathbf{0}$ positive semidefinite ($> \mathbf{0}$ positive definite). The following result about the maximal solution of NME was given in Refs. [11,12]:

Theorem 2.1. *Eq.* (2.5) *has a symmetric positive definite solution if and only if* $\psi(\lambda) = \lambda\mathbf{A} + \mathbf{Q} + \lambda^{-1}\mathbf{A}^\top$ *is regular* (*i.e., the determinant of* $\psi(\lambda)$ *is not identically zero*) *and* $\psi(\lambda) \geqslant \mathbf{0}$ *for all* λ *on the unit circle. Furthermore, if Eq.* (2.5) *has a symmetric positive definite solution, it has a maximal solution* $\mathbf{Z}_+$ *with* $\rho(\mathbf{Z}_+^{-1}\mathbf{A}) \leqslant 1$. *For any other symmetric positive definite solution* $\mathbf{Z}$, *it holds* $\rho(\mathbf{Z}^{-1}\mathbf{A}) > 1$. *Moreover,* $\rho(\mathbf{Z}_+^{-1}\mathbf{A}) < 1$ *if and only if* $\psi(\lambda) > \mathbf{0}$ *for all* λ *on the unit circle.*

Two theorems in Ref. [1] show the relations between the NME (2.5) and the QEP (1.2).

Theorem 2.2 (*Guo [1, Theorem 6]*). *The QEP* (1.2) *has no eigenvalues on the imaginary axis if and only if* $\psi(\lambda) = \lambda\mathbf{A} + \mathbf{Q} + \lambda^{-1}\mathbf{A}^\top > \mathbf{0}$ *for all* λ *on the unit circle, where the matrices* $\mathbf{A}, \mathbf{Q}$ *are given in Eq.* (2.4).

Theorem 2.3 (*Guo [1, Theorem 7]*). *Assume that* $\psi(\lambda) \geqslant \mathbf{0}$ *for all* λ *on the unit circle and let* $\mathbf{Z}_+$ *be the maximal solution of Eq.* (2.5). *Then the eigenvalue of* $-\mathbf{Z}_+^{-1}\mathbf{A}$ *are precisely the eigenvalues of* $\phi(\lambda) = \lambda^2\mathbf{A}^\top + \lambda\mathbf{Q} + \mathbf{A}$ *inside or on the unit circle, with the same partial multiplicities for each eigenvalue inside the unit circle and with half of the partial multiplicities for each unimodular eigenvalue.*

Guo [1] also proved that μ_i is an eigenvalue of $\phi(\mu)$ if and only if $\lambda_i = (1+\mu_i)/(1-\mu_i)$ is an eigenvalue of $G(\lambda)$, and the partial multiplicities of μ_i and λ_i are invariant. Together with these two theorems, we can see that all eigenvalues of the QEP (1.2) can be found by solving the eigenvalues of $-\mathbf{Z}_+^{-1}\mathbf{A}$, where $\mathbf{Z}_+$ is the maximal solution of Eq. (2.5), under the assumption that $\psi(\lambda)\geqslant\mathbf{0}$ for all λ on the unit circle. Moreover, if $\mathbf{x}_i$ and $\mathbf{y}_i$ are, respectively, the right and left eigenvectors of $-\mathbf{Z}_+^{-1}\mathbf{A}$ corresponding to μ_i, then $\mathbf{x}_i$ and $\mathbf{y}_i$ are, respectively, the right and left eigenvectors of $\mathbf{S} = (\mathbf{I}-\mathbf{Z}_+^{-1}\mathbf{A})(\mathbf{I}+\mathbf{Z}_+^{-1}\mathbf{A})^{-1}$ corresponding to $\lambda_i = (1+\mu_i)/(1-\mu_i)$. Hence, $\mathbf{x}_i$ and $(\lambda_i\mathbf{M}+\mathbf{M}\mathbf{S}+\mathbf{G})^{-\top}\bar{\mathbf{y}}_i$ are the eigenvectors of the QEP corresponding to $\pm\lambda_i$.

The problem is now reduced to computing the maximal solution of Eq. (2.5) efficiently. Several numerical methods have been proposed, including the cyclic reduction method [8] and the structure-preserving doubling algorithm [9]. These algorithms are similar, but the convergence analysis of the structure-preserving doubling algorithm is simpler. The structure-preserving doubling algorithm is as follows.

Algorithm 2.1.

$$\begin{aligned}
&\mathbf{A}_0 = \mathbf{A}, \quad \mathbf{Q}_0 = \mathbf{Q}, \quad \mathbf{P}_0 = \mathbf{0},\\
&\mathbf{A}_{k+1} = \mathbf{A}_k(\mathbf{Q}_k - \mathbf{P}_k)^{-1}\mathbf{A}_k,\\
&\mathbf{Q}_{k+1} = \mathbf{Q}_k - \mathbf{A}_k^{\top}(\mathbf{Q}_k - \mathbf{P}_k)^{-1}\mathbf{A}_k,\\
&\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{A}_k(\mathbf{Q}_k - \mathbf{P}_k)^{-1}\mathbf{A}_k^{\top}.
\end{aligned}$$

One step of Algorithm 2.1 requires $\frac{7}{3}n^3$ flops, similar to that of the cyclic reduction method [8].

To ensure that the iteration in Algorithm 2.1 is well-defined, $\mathbf{Q}_k - \mathbf{P}_k$ must be symmetric positive definite for all k. This is guaranteed by the theorem in Ref. [9], which describes the convergence of Algorithm 2.1.

Theorem 2.4 (*Lin and Xu [9]*). *Assume that* $\mathbf{Z}>\mathbf{0}$ *is a solution of Eq.* (2.5), *and let* $\mathbf{R} = \mathbf{Z}^{-1}\mathbf{A}$. *Then the matrix sequences* $\{\mathbf{A}_k\}, \{\mathbf{Q}_k\}$ *and* $\{\mathbf{P}_k\}$ *generated by Algorithm* 2.1 *are well-defined and satisfy*:

1. $\mathbf{A}_k = (\mathbf{Z}-\mathbf{P}_k)\mathbf{R}^{2^k}$;
2. $\mathbf{0}\leqslant\mathbf{P}_k\leqslant\mathbf{P}_{k+1}<\mathbf{Z}$ *and* $\mathbf{Q}_k - \mathbf{P}_k = (\mathbf{Z}-\mathbf{P}_k) + \mathbf{A}_k^{\top}(\mathbf{Z}-\mathbf{P}_k)^{-1}\mathbf{A}_k > \mathbf{0}$;
3. $\mathbf{Z}\leqslant\mathbf{Q}_{k+1}\leqslant\mathbf{Q}_k\leqslant\mathbf{Q}$ *and* $\mathbf{Q}_k - \mathbf{Z} = (\mathbf{R}^{\top})^{2^k}(\mathbf{Z}-\mathbf{P}_k)\mathbf{R}^{2^k}\leqslant(\mathbf{R}^{\top})^{2^k}\mathbf{Z}\mathbf{R}^{2^k}$.

Moreover, if the maximal solution $\mathbf{Z}_+$ *satisfies* $\rho(\mathbf{S}_+)<1$, *where* $\mathbf{S}_+ = \mathbf{Z}_+^{-1}\mathbf{A}$, *we have*

$$\begin{aligned}
&\|\mathbf{A}_k\|_2 \leqslant \|\mathbf{Z}_+\|_2\|\mathbf{S}_+^{2^k}\|_2 \to 0 \quad \text{as } k\to\infty,\\
&\|\mathbf{Z}_+ - \mathbf{Q}_k\|_2 \leqslant \|\mathbf{Z}_+\|_2\|\mathbf{S}_+^{2^k}\|_2^2 \to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Theorem 2.4 shows that Algorithm 2.1 converges quadratically when no eigenvalues of $\mathbf{Z}_+^{-1}\mathbf{A}$ lies on the unit circle. When $\rho(\mathbf{Z}_+^{-1}\mathbf{A}) = 1$, Chu et al. [10] proved the following result.

Theorem 2.5. *Assume that* $\mathbf{Z}_+$ *is the maximal solution of Eq.* (2.5), *and* $\rho(\mathbf{Z}_+^{-1}\mathbf{A}) = 1$. *If the partial multiplicities of* $\mathbf{Z}_+^{-1}\mathbf{A}$ *associated with the unimodular eigenvalues are all even, then the matrix sequence* $\{\mathbf{Q}_k\}$ *generated by Algorithm* 2.1 *converges to* $\mathbf{Z}_+$ *with convergence rate* 1/2.

3. Eigenvalue shifting

Theorem 2.2, together with Theorems 2.1 and 2.4, show that if the QEP (1.2) has no eigenvalues on the imaginary axis, the matrix sequence $\{Q_k\}$ generated by Algorithm 2.1 converges quadratically to the maximal solution $\mathbf{Z}_+$ of the NME (2.5). However, if the QEP (1.2) has eigenvalues on the imaginary axis, then $\psi(\lambda) = \lambda^2\mathbf{A}^{\top} + \lambda\mathbf{Q} + \mathbf{A}$ has eigenvalues on the unit circle (because of the Cayley transformation involved). Hence if the conditions in Theorem 2.5 hold, Algorithm 2.1 still converges, but with a linear or slower convergence rate. So it is desirable to shift purely imaginary eigenvalues away from the imaginary axis while keeping the remaining eigenpairs invariant.

Chu et al. [13] discussed model updating problems with no spill-over for quadratic pencils, which incorporates the original quadratic model with some measured data. The updated model matches the measured data preserving part of original eigenstruture. Based on this approach, we transform the original gyroscopic system $(\mathbf{M},\mathbf{G},\mathbf{K})$ to a new gyroscopic system

$$(\tilde{\mathbf{M}},\tilde{\mathbf{G}},\tilde{\mathbf{K}}) \equiv (\mathbf{M}+\Delta\mathbf{M},\mathbf{G}+\Delta\mathbf{G},\mathbf{K}+\Delta\mathbf{K}),$$

with $\Delta\mathbf{M}=\Delta\mathbf{M}^{\top},\Delta\mathbf{G}=-\Delta\mathbf{G}^{\top},\Delta\mathbf{K}=\Delta\mathbf{K}^{\top}$ such that the original eigenvalues on the imaginary axis are shifted away from the imaginary axis while preserving the other part of the original eigenstructure.

It is well known that for $G(\lambda)$, there exist an $n\times 2n$ eigenvector matrix $\mathbf{X}$ and an $2n\times 2n$ eigenvalue matrix $\mathbf{J}$ in Jordan canonical form such that $\begin{bmatrix}\mathbf{J}\\ \mathbf{XJ}\end{bmatrix}$ is nonsingular and

$$\mathbf{MXJ}^2+\mathbf{GXJ}+\mathbf{KX}=\mathbf{0}. \tag{3.1}$$

Here $(\mathbf{X},\mathbf{J})$ is referred to as a Jordan pair of $G(\lambda)$. Suppose that $\mathbf{J}$ and $\mathbf{X}$ are partitioned as $\mathbf{J}=\mathrm{diag}(\mathbf{J}_1,\mathbf{J}_2)$ and $\mathbf{X}=[\mathbf{X}_1\ \mathbf{X}_2]$, where $\mathbf{X}_1$ consists of the generalized eigenvectors corresponding to $\mathbf{J}_1$. The following theorem gives an orthogonalization relationship between eigenvalue matrices and eigenvector matrices.

Theorem 3.1. *Consider* $G(\lambda)=\lambda^2\mathbf{M}+\lambda\mathbf{G}+\mathbf{K}$ *with* $\mathbf{M}=\mathbf{M}^{\top}$ *being positive definite*, $\mathbf{G}=-\mathbf{G}^{\top},\mathbf{K}=\mathbf{K}^{\top}$. *Let* $(\mathbf{X}_1,\mathbf{J}_1)$ *and* $(\mathbf{X}_2,\mathbf{J}_2)$ *be defined as above. If* $\sigma(\mathbf{J}_1)\cap\sigma(-\mathbf{J}_2)=\emptyset$, *then we have*

$$\text{(i) } \mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{MX}_2\mathbf{J}_2+\mathbf{X}_1^{\top}\mathbf{KX}_2=\mathbf{0}, \tag{3.2}$$

$$\text{(ii) } \mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{GX}_2\mathbf{J}_2-\mathbf{X}_1^{\top}\mathbf{KX}_2\mathbf{J}_2+\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{KX}_2=\mathbf{0}, \tag{3.3}$$

$$\text{(iii) } \mathbf{X}_1^{\top}\mathbf{MX}_2\mathbf{J}_2-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{MX}_2+\mathbf{X}_1^{\top}\mathbf{GX}_2=\mathbf{0}. \tag{3.4}$$

Proof. We first prove Eq. (3.4). Obviously, $(\mathbf{X}_1,\mathbf{J}_1)$ and $(\mathbf{X}_2,\mathbf{J}_2)$ satisfy

$$\mathbf{MX}_2\mathbf{J}_2^2+\mathbf{GX}_2\mathbf{J}_2+\mathbf{KX}_2=\mathbf{0}, \tag{3.5}$$

$$\mathbf{J}_1^{2\top}\mathbf{X}_1^{\top}\mathbf{M}-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{G}+\mathbf{X}_1^{\top}\mathbf{K}=\mathbf{0}. \tag{3.6}$$

Multiplying Eqs. (3.5) and (3.6) by $\mathbf{X}_1^{\top}$ and $\mathbf{X}_2$ on the left and the right, respectively, and eliminating $\mathbf{X}_1^{\top}\mathbf{KX}_2$ we obtain

$$\mathbf{X}_1^{\top}\mathbf{MX}_2\mathbf{J}_2^2+\mathbf{X}_1^{\top}\mathbf{GX}_2\mathbf{J}_2=\mathbf{J}_1^{2\top}\mathbf{X}_1^{\top}\mathbf{MX}_2-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{GX}_2. \tag{3.7}$$

Rewrite Eq. (3.7) into

$$\begin{aligned}&(\mathbf{X}_1^{\top}\mathbf{MX}_2\mathbf{J}_2-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{MX}_2+\mathbf{X}_1^{\top}\mathbf{GX}_2)\mathbf{J}_2\\ &\quad+\mathbf{J}_1^{\top}(\mathbf{X}_1^{\top}\mathbf{MX}_2\mathbf{J}_2-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{MX}_2+\mathbf{X}_1^{\top}\mathbf{GX}_2)=\mathbf{0}.\end{aligned} \tag{3.8}$$

By the assumption $\sigma(\mathbf{J}_1)\cap\sigma(-\mathbf{J}_2)=\emptyset$ and applying the Lyapunov Theorem to Eq. (3.8) we get

$$\mathbf{X}_1^{\top}\mathbf{MX}_2\mathbf{J}_2-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{MX}_2+\mathbf{X}_1^{\top}\mathbf{GX}_2=\mathbf{0}.$$

The relations of Eqs. (3.2) and (3.3) can be shown in a similar way by eliminating the '**G**-term' and '**M**-term' in Eqs. (3.5) and (3.6), respectively. □

Now we assume that $\mathbf{J}_1$ contains all eigenvalues lying on the imaginary axis. To avoid complex arithmetic, we assume that $\mathbf{J}_1\in\mathbb{R}^{k\times k}$ and $\mathbf{X}_1\in\mathbb{R}^{n\times k}$ are of the forms

$$\mathbf{J}_1=\mathrm{diag}(\mathbf{P}_1,\ldots,\mathbf{P}_s;\mathbf{N}_1,\ldots,\mathbf{N}_t),\quad \mathbf{X}_1=[\mathbf{Y}_1,\ldots,\mathbf{Y}_s;\mathbf{Z}_1,\ldots,\mathbf{Z}_t], \tag{3.9}$$

with

$$\mathbf{P}_j = \begin{bmatrix} \mathbf{\Lambda}_j & \mathbf{I} & & \\ & \ddots & \ddots & \\ & & \mathbf{\Lambda}_j & \mathbf{I} \\ & & & \mathbf{\Lambda}_j \end{bmatrix} \in \mathbb{R}^{2m_j \times 2m_j}, \quad \mathbf{\Lambda}_j = \begin{bmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{bmatrix}, \tag{3.10}$$

$$\mathbf{Y}_j = [\mathbf{y}_{1R}^{(j)}, \ \mathbf{y}_{1I}^{(j)}, \ \ldots, \ \mathbf{y}_{m_jR}^{(j)}, \ \mathbf{y}_{m_jI}^{(j)}] \in \mathbb{R}^{n \times 2m_j}, \quad j = 1, \ldots, s$$

and

$$\mathbf{N}_j = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \in \mathbb{R}^{n_j \times n_j}, \tag{3.11}$$

$$\mathbf{Z}_j = [\mathbf{z}_1, \ \ldots, \ \mathbf{z}_{n_j}] \in \mathbb{R}^{n \times n_j}, \quad j = 1, \ldots, t.$$

Here $\mathbf{P}_j$ contains the purely imaginary eigenvalues $\pm \mathrm{i}\alpha_j$ with partial multiplicities m_j and m_j, and $\mathbf{y}_{1R}^{(j)} \pm \mathrm{i}\mathbf{y}_{1I}^{(j)}, \ldots, \mathbf{y}_{m_jR}^{(j)} \pm \mathrm{i}\mathbf{y}_{m_jI}^{(j)}$ are the associated generalized eigenvectors ($j = 1, \ldots, s$); $\mathbf{N}_j$ contains the zero eigenvalues with partial multiplicities n_j, and $\mathbf{z}_1, \ldots, \mathbf{z}_{n_j}$ are the associated generalized eigenvectors ($j = 1, \ldots, t$). We say that $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are the generalized eigenvectors corresponding to the eigenvalue λ_0 provided that

$$\begin{aligned} G(\lambda_0)\mathbf{x}_1 &= \mathbf{0}, \\ G(\lambda_0)\mathbf{x}_2 + (2\lambda_0\mathbf{M} + \mathbf{G})\mathbf{x}_1 &= \mathbf{0}, \\ G(\lambda_0)\mathbf{x}_3 + (2\lambda_0\mathbf{M} + \mathbf{G})\mathbf{x}_2 + \mathbf{M}\mathbf{x}_1 &= \mathbf{0}, \\ &\vdots \\ G(\lambda_0)\mathbf{x}_m + (2\lambda_0\mathbf{M} + \mathbf{G})\mathbf{x}_{m-1} + \mathbf{M}\mathbf{x}_{m-2} &= \mathbf{0}. \end{aligned} \tag{3.12}$$

On the other hand, $(\mathbf{X}_2, \mathbf{J}_2)$ is the part of eigenstructure with eigenvalues having nonzero real parts which we want to preserve. Hence, the triplet of $(\Delta\mathbf{M}, \Delta\mathbf{G}, \Delta\mathbf{K})$ satisfies

$$(\mathbf{M} + \Delta\mathbf{M})\mathbf{X}_2\mathbf{J}_2^2 + (\mathbf{G} + \Delta\mathbf{G})\mathbf{X}_2\mathbf{J}_2 + (\mathbf{K} + \Delta\mathbf{K})\mathbf{X}_2 = \mathbf{0}.$$

From Eq. (3.5) it implies

$$\Delta\mathbf{M}\mathbf{X}_2\mathbf{J}_2^2 + \Delta\mathbf{G}\mathbf{X}_2\mathbf{J}_2 + \Delta\mathbf{K}\mathbf{X}_2 = \mathbf{0}. \tag{3.13}$$

The next theorem gives a sufficient condition for Eq. (3.13).

Theorem 3.2. *Let*

$$\Delta\mathbf{M} = \mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}, \tag{3.14}$$

$$\Delta\mathbf{G} = \mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{G} + \mathbf{G}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M} - \mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M} + \mathbf{M}\mathbf{X}_1\mathbf{J}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}, \tag{3.15}$$

$$\Delta\mathbf{K} = (\mathbf{M}\mathbf{X}_1\mathbf{J}_1 + \mathbf{G}\mathbf{X}_1)\mathbf{\Phi}(-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M} + \mathbf{X}_1^{\top}\mathbf{G}). \tag{3.16}$$

Then for any real symmetric matrix $\mathbf{\Phi}$, $(\Delta\mathbf{M}, \Delta\mathbf{G}, \Delta\mathbf{K})$ *defined by Eqs.* (3.14)–(3.16) *is a solution to Eq.* (3.13) *satisfying* $\Delta\mathbf{M} = \Delta\mathbf{M}^{\top}, \Delta\mathbf{G} = -\Delta\mathbf{G}^{\top}, \Delta\mathbf{K} = \Delta\mathbf{K}^{\top}$.

Proof. It is easy to verify that if $\mathbf{\Phi}$ is symmetric, $(\Delta\mathbf{M}, \Delta\mathbf{G}, \Delta\mathbf{K})$ defined by Eqs. (3.14)–(3.16) satisfies $\Delta\mathbf{M} = \Delta\mathbf{M}^{\top}, \Delta\mathbf{G} = -\Delta\mathbf{G}^{\top}, \Delta\mathbf{K} = \Delta\mathbf{K}^{\top}$.

Since $\mathbf{J}_1$ contains all eigenvalues on the imaginary axis, the condition $\sigma(\mathbf{J}_1) \cap \sigma(-\mathbf{J}_2) = \emptyset$ is automatically satisfied, thus Eq. (3.4) holds. By direct calculations using Eqs. (3.14)–(3.16), and

from Eq. (3.4), we have

$$\begin{aligned}
&\Delta\mathbf{M}\mathbf{X}_2\mathbf{J}_2^2+\Delta\mathbf{G}\mathbf{X}_2\mathbf{J}_2+\Delta\mathbf{K}\mathbf{X}_2\\
&\quad=\mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_2\mathbf{J}_2^2+\mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{G}\mathbf{X}_2\mathbf{J}_2-\mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_2\mathbf{J}_2+\mathbf{G}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_2\mathbf{J}_2\\
&\qquad+\mathbf{M}\mathbf{X}_1\mathbf{J}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_2\mathbf{J}_2+(\mathbf{M}\mathbf{X}_1\mathbf{J}_1+\mathbf{G}\mathbf{X}_1)\mathbf{\Phi}(-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}+\mathbf{X}_1^{\top}\mathbf{G})\mathbf{X}_2\\
&\quad=\mathbf{M}\mathbf{X}_1\mathbf{\Phi}(\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_2\mathbf{J}_2-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_2+\mathbf{X}_1^{\top}\mathbf{G}\mathbf{X}_2)\mathbf{J}_2\\
&\qquad+(\mathbf{M}\mathbf{X}_1\mathbf{J}_1+\mathbf{G}\mathbf{X}_1)\mathbf{\Phi}(\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_2\mathbf{J}_2-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_2+\mathbf{X}_1^{\top}\mathbf{G}\mathbf{X}_2)\\
&\quad=\mathbf{0}. \qquad \square
\end{aligned} \tag{3.17}$$

Remark. (i) In Theorem 3.2, we use Eq. (3.4) to show that Eq. (3.13) holds. Similarly, we can use Eq. (3.2) to show that $(\Delta\mathbf{M},\Delta\mathbf{G},\Delta\mathbf{K})$ of the forms

$$\Delta\mathbf{M}=\mathbf{M}\mathbf{X}_1\mathbf{J}_1\mathbf{\Phi}\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M},$$

$$\Delta\mathbf{G}=\mathbf{M}\mathbf{X}_1\mathbf{J}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{K}-\mathbf{K}\mathbf{X}_1\mathbf{\Phi}\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M},$$

$$\Delta\mathbf{K}=-\mathbf{K}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{K}$$

with $\mathbf{\Phi}$ being symmetric, also satisfies (3.13). Unfortunately, if $\mathbf{K}$ is singular, then $\mathbf{K}+\Delta\mathbf{K}=(\mathbf{I}-\mathbf{K}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top})\mathbf{K}$ is still singular. Consequently, the zeroes eigenvalues of the original gyroscopic system are invariant in the new updated gyroscopic system. This is not the case what we required.

(ii) In Theorem 3.2 we show that the eigenstructure $(\mathbf{X}_2,\mathbf{J}_2)$ of the original model $(\mathbf{M},\mathbf{G},\mathbf{K})$ is preserved in the new updated model when the incremental triplet $(\Delta\mathbf{M},\Delta\mathbf{G},\Delta\mathbf{K})$ is given by Eqs. (3.14)–(3.16).

The following theorem shows that the relation of eigenvalues between the original model and the updated model.

Theorem 3.3. *Let* $(\Delta\mathbf{M},\Delta\mathbf{G},\Delta\mathbf{K})$ *be given by Eqs.* (3.14)–(3.16). *Define* $(\tilde{\mathbf{M}},\tilde{\mathbf{G}},\tilde{\mathbf{K}})\equiv(\mathbf{M}+\Delta\mathbf{M},\mathbf{G}+\Delta\mathbf{G},\mathbf{K}+\Delta\mathbf{K})$. *Then the eigenvalues of* $\tilde{G}(\lambda)=\lambda^2\tilde{\mathbf{M}}+\lambda\tilde{\mathbf{G}}+\tilde{\mathbf{K}}$ *are those of* $\mathbf{J}_2$ *together with those of the matrix pencil*

$$(\mathbf{J}_1+\mathbf{\Phi}\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1-\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{G}\mathbf{X}_1,\mathbf{I}+\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1). \tag{3.18}$$

Proof. From $\mathbf{M}\mathbf{X}_1\mathbf{J}_1^2+\mathbf{G}\mathbf{X}_1\mathbf{J}_1+\mathbf{K}\mathbf{X}_1=\mathbf{0}$, we have

$$G(\lambda)\mathbf{X}_1=(\lambda^2\mathbf{M}+\lambda\mathbf{G}+\mathbf{K})\mathbf{X}_1=(\lambda\mathbf{M}\mathbf{X}_1+\mathbf{M}\mathbf{X}_1\mathbf{J}_1+\mathbf{G}\mathbf{X}_1)(\lambda\mathbf{I}-\mathbf{J}_1)$$

which implies that

$$(\lambda\mathbf{M}\mathbf{X}_1+\mathbf{M}\mathbf{X}_1\mathbf{J}_1+\mathbf{G}\mathbf{X}_1)=G(\lambda)\mathbf{X}_1(\lambda\mathbf{I}-\mathbf{J}_1)^{-1}.$$

From Eqs. (3.14)–(3.16), we have

$$\begin{aligned}
\tilde{G}(\lambda)&=\lambda^2(\mathbf{M}+\Delta\mathbf{M})+\lambda(\mathbf{G}+\Delta\mathbf{G})+(\mathbf{K}+\Delta\mathbf{K})\\
&=G(\lambda)+\lambda^2\mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}+\lambda(\mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{G}+\mathbf{G}\mathbf{X}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}-\mathbf{M}\mathbf{X}_1\mathbf{\Phi}\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\\
&\quad+\mathbf{M}\mathbf{X}_1\mathbf{J}_1\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M})+(\mathbf{M}\mathbf{X}_1\mathbf{J}_1+\mathbf{G}\mathbf{X}_1)\mathbf{\Phi}(-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}+\mathbf{X}_1^{\top}\mathbf{G})\\
&=G(\lambda)+(\lambda\mathbf{M}\mathbf{X}_1+\mathbf{M}\mathbf{X}_1\mathbf{J}_1+\mathbf{G}\mathbf{X}_1)\mathbf{\Phi}(\lambda\mathbf{X}_1^{\top}\mathbf{M}-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}+\mathbf{X}_1^{\top}\mathbf{G})\\
&=G(\lambda)+G(\lambda)\mathbf{X}_1(\lambda\mathbf{I}-\mathbf{J}_1)^{-1}\mathbf{\Phi}(\lambda\mathbf{X}_1^{\top}\mathbf{M}-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}+\mathbf{X}_1^{\top}\mathbf{G})\\
&=G(\lambda)(\mathbf{I}+\mathbf{X}_1(\lambda\mathbf{I}-\mathbf{J}_1)^{-1}\mathbf{\Phi}(\lambda\mathbf{X}_1^{\top}\mathbf{M}-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}+\mathbf{X}_1^{\top}\mathbf{G})).
\end{aligned}$$

Hence, with the help of $\det(\mathbf{I}+\mathbf{RS})=\det(\mathbf{I}+\mathbf{SR})$, where $\mathbf{R}\in\mathbb{C}^{n\times m}$ and $\mathbf{S}\in\mathbb{C}^{m\times n}$, we have

$$\begin{aligned}
&\det\big(\tilde{G}(\lambda)\big)\\
&=\det(G(\lambda))\det\big(\mathbf{I}+\mathbf{X}_1(\lambda\mathbf{I}-\mathbf{J}_1)^{-1}\mathbf{\Phi}(\lambda\mathbf{X}_1^{\top}\mathbf{M}-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}+\mathbf{X}_1^{\top}\mathbf{G})\big)\\
&=\det(G(\lambda))\det\big(\mathbf{I}+(\lambda\mathbf{I}-\mathbf{J}_1)^{-1}\mathbf{\Phi}(\lambda\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1+\mathbf{X}_1^{\top}\mathbf{G}\mathbf{X}_1)\big)\\
&=\det(G(\lambda))\det\big((\lambda\mathbf{I}-\mathbf{J}_1)^{-1}\big)\det\big(\lambda\mathbf{I}-\mathbf{J}_1+\mathbf{\Phi}(\lambda\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1-\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1+\mathbf{X}_1^{\top}\mathbf{G}\mathbf{X}_1)\big)\\
&=\det(G(\lambda))\det\big((\lambda\mathbf{I}-\mathbf{J}_1)^{-1}\big)\det\big(\lambda(\mathbf{I}+\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1)-(\mathbf{J}_1+\mathbf{\Phi}\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1-\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{G}\mathbf{X}_1)\big).
\end{aligned}$$

Since $G(\lambda)$ has the eigenvalues of $\mathbf{J}_1$, the above equality shows that $\tilde{G}(\lambda)$ and $G(\lambda)$ share the same spectrum, except that the eigenvalues of $\mathbf{J}_1$ are replaced by those in

$$(\mathbf{J}_1+\mathbf{\Phi}\mathbf{J}_1^{\top}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1-\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{G}\mathbf{X}_1,\mathbf{I}+\mathbf{\Phi}\mathbf{X}_1^{\top}\mathbf{M}\mathbf{X}_1). \qquad \square$$

Remark. (i) Theorem 3.3 shows that $\tilde{G}(\lambda)$ keeps the eigenvalues of $G(\lambda)$ with nonzero real parts and changes all purely imaginary eigenvalues of $G(\lambda)$ to those eigenvalues of the pencil in Eq. (3.18). The new updated quadratic pencil $\tilde{G}(\lambda)$, generically, has no eigenvalues on the imaginary axis for a randomly chosen symmetric matrix $\mathbf{\Phi}$.

(ii) If $\mathbf{\Phi}$ is chosen to be positive semidefinite, then $\Delta\mathbf{M}$ defined in Eq. (3.14) is positive semidefinite, and $\Delta\mathbf{K}$ in Eq. (3.16) is negative semidefinite, which ensures that the corresponding $\tilde{\mathbf{Q}}=2(\tilde{\mathbf{M}}-\tilde{\mathbf{K}})$ is still positive definite.

For the computation of $(\mathbf{X}_1,\mathbf{J}_1)$, we apply the Newton's method [14] to find a purely imaginary eigenvalue $\mathrm{i}\omega$ of $G(\lambda)$ and the associated eigenvector $\mathbf{y}+\mathrm{i}\mathbf{z}$ satisfying $G(\mathrm{i}\omega)(\mathbf{y}+\mathrm{i}\mathbf{z})=\mathbf{0}$. This eigenequation can be reformulated by the nonlinear equation in $\boldsymbol{\zeta}\equiv[\mathbf{y}^{\top},\mathbf{z}^{\top},\omega]^{\top}$ with an additional normalization condition:

$$T\left(\begin{bmatrix}\mathbf{y}\\ \mathbf{z}\\ \omega\end{bmatrix}\right)\equiv T(\boldsymbol{\zeta})\equiv\begin{bmatrix}T_1(\boldsymbol{\zeta})\\ T_2(\boldsymbol{\zeta})\\ T_3(\boldsymbol{\zeta})\end{bmatrix}=\begin{bmatrix}(-\omega^2\mathbf{M}+\mathbf{K})\mathbf{y}+\omega\mathbf{G}\mathbf{z}\\ -\omega\mathbf{G}\mathbf{y}+(-\omega^2\mathbf{M}+\mathbf{K})\mathbf{z}\\ \mathbf{y}^{\top}\mathbf{y}+\mathbf{z}^{\top}\mathbf{z}-1\end{bmatrix}=\mathbf{0}. \tag{3.19}$$

We differentiate and obtain the Fréchet derivative of Eq. (3.19)

$$T'\left(\begin{bmatrix}\mathbf{y}\\ \mathbf{z}\\ \omega\end{bmatrix}\right)=\begin{bmatrix}-\omega^2\mathbf{M}+\mathbf{K} & \omega\mathbf{G} & -2\omega\mathbf{M}\mathbf{y}+\mathbf{G}\mathbf{z}\\ -\omega\mathbf{G} & -\omega^2\mathbf{M}+\mathbf{K} & -\mathbf{G}\mathbf{y}-2\omega\mathbf{M}\mathbf{z}\\ 2\mathbf{y}^{\top} & 2\mathbf{z}^{\top} & 0\end{bmatrix}. \tag{3.20}$$

Newton's method now amounts to solving a $(2n+1)$-dimensional linear system in each iteration with an initial vector $\boldsymbol{\zeta}_0$:

$$T'(\boldsymbol{\zeta}_k)(\boldsymbol{\zeta}_{k+1}-\boldsymbol{\zeta}_k)=-T(\boldsymbol{\zeta}_k), \tag{3.21}$$

where $\boldsymbol{\zeta}_k\equiv[\mathbf{y}_k^{\top},\mathbf{z}_k^{\top},\omega^{(k)}]^{\top}$. The Newton's method (3.21) is equivalent to the inverse iteration which is more conveniently formulated as

$$\begin{bmatrix}-\omega^{(k)^2}\mathbf{M}+\mathbf{K} & \omega^{(k)}\mathbf{G}\\ -\omega^{(k)}\mathbf{G} & -\omega^{(k)^2}\mathbf{M}+\mathbf{K}\end{bmatrix}\begin{bmatrix}\mathbf{u}_{k+1}\\ \mathbf{v}_{k+1}\end{bmatrix}=\begin{bmatrix}-2\omega^{(k)}\mathbf{M}\mathbf{y}_k+\mathbf{G}\mathbf{z}_k\\ -\mathbf{G}\mathbf{y}_k-2\omega^{(k)}\mathbf{M}\mathbf{z}_k\end{bmatrix}, \tag{3.22a}$$

$$\omega^{(k+1)}=\omega^{(k)}-\frac{1}{\mathbf{y}_k^{\top}\mathbf{u}_{k+1}+\mathbf{z}_k^{\top}\mathbf{v}_{k+1}}, \tag{3.22b}$$

$$\begin{bmatrix}\mathbf{y}_{k+1}\\ \mathbf{z}_{k+1}\end{bmatrix}=\frac{1}{\sqrt{\mathbf{u}_{k+1}^{\top}\mathbf{u}_{k+1}+\mathbf{v}_{k+1}^{\top}\mathbf{v}_{k+1}}}\begin{bmatrix}\mathbf{u}_{k+1}\\ \mathbf{v}_{k+1}\end{bmatrix}. \tag{3.22c}$$

The symmetric linear system (3.22a) can be solved by the $\mathrm{LDL}^{\top}$-factorization (see e.g. Ref. [15]) which requires $\frac{8}{3}n^3$ flops.

Once the inverse iteration (3.22) converges to $[\mathbf{y}^\top, \mathbf{z}^\top, \omega]^\top$, we can apply Eq. (3.12) by setting $\lambda_0 = \mathrm{i}\omega$ and $\mathbf{x}_1 = \mathbf{y} + \mathrm{i}\mathbf{z}$ to compute the Jordan basis $\mathbf{X}_{1\omega}$ and the Jordan block $\mathbf{J}_{1\omega}$ corresponding to $\mathrm{i}\omega$. The following algorithm computes the Jordan basis $\mathbf{X}_1$ and the Jordan block $\mathbf{J}_1$ corresponding to purely imaginary eigenvalues of $G(\lambda)$. For convenience, we suppose that the geometrical multiplicity of each nonzero purely imaginary eigenvalues is one.

Algorithm 3.1 (*Initiation*). $\mathbf{J}_1 = \emptyset, \mathbf{X}_1 = \emptyset, T = \emptyset, \mathbf{U} = \emptyset, \mathbf{K}_1 := \mathbf{K}, R \gg 1, \omega^{(0)} = \varepsilon > 0, \mathrm{Tol} = \textit{tolerance}$;

1. If $\mathbf{K}_1$ is singular or nearly singular, then
   1.1. Compute $\|\mathbf{K}_1\mathbf{x}_1\| \leqslant \mathrm{Tol}$ with $\mathbf{x}_1 \perp \mathbf{U}$ and $\|\mathbf{x}_1\| = 1$;
   1.2. Set $\omega := 0, \mathbf{U} := [\mathbf{U}, \mathbf{x}_1], \mathbf{K}_1 := \mathbf{K}_1 - \mathbf{U}\mathbf{U}^\top$, go to 5; else
2. If $\omega^{(0)} > R$, stop; else
3. Choose $\mathbf{y}^{(0)} + \mathrm{i}\mathbf{z}^{(0)} \perp \mathbf{U}$ with $\|\mathbf{y}^{(0)} + \mathrm{i}\mathbf{z}^{(0)}\| = 1$;
4. Compute Eq. (3.22) until convergence; Set $\omega \leftarrow \omega^{(k+1)}, \mathbf{x}_1 \leftarrow \mathbf{y}^{(k+1)} + \mathrm{i}\mathbf{z}^{(k+1)}, \mathbf{U} = [\mathbf{U}, \mathbf{x}_1]$;
5. $j = 2, \mathbf{x}_0 = \mathbf{0}, \mathbf{X}_{1\omega} = [\mathbf{x}_1], \mathbf{J}_{1\omega} = [\mathrm{i}\omega]$;
   5.1. Compute $\|\xi_1^\top G(\mathrm{i}\omega)\| \leqslant \mathrm{Tol}$ with $\|\xi_1\| = 1$;
   5.2. Compute $\mathbf{b} = -(2\mathrm{i}\omega\mathbf{M} + \mathbf{G})\mathbf{x}_{j-1} - \mathbf{M}\mathbf{x}_{j-2}$,

   If $\xi_1^\top\mathbf{b} \gg \mathrm{Tol}$, then go to 6; else
   Solve $G(\mathrm{i}\omega)\mathbf{x}_j = \mathbf{b}$,

   $\mathbf{X}_{1\omega} := [\mathbf{X}_{1\omega}, \mathbf{x}_j], \mathbf{J}_{1\omega} := \begin{bmatrix} \mathbf{J}_{1\omega} & \mathbf{e}_* \\ \mathbf{0} & \mathrm{i}\omega \end{bmatrix}$, where $\mathbf{e}_* = [0, \ldots, 0, 1]^\top$,

   $j \leftarrow j + 1, \mathbf{x}_{j-2} \leftarrow \mathbf{x}_{j-1}, \mathbf{x}_{j-1} \leftarrow \mathbf{x}_j$, repeat 5.2;
6. $\mathbf{X}_1 := [\mathbf{X}_1, \mathbf{X}_{1\omega}], \mathbf{J}_1 := \mathbf{J}_1 \oplus \mathbf{J}_{1\omega}, T := T \cup \{\mathrm{i}\omega\}$;

   If $\mathbf{K}_1$ is singular or nearly singular go to 1.1; else
   $\omega^{(0)} \leftarrow \omega^{(0)} + 2|\omega - \omega^{(0)}|$; go to 2.

Remark. (i) The shift-strategy $\omega^{(0)} := \omega^{(0)} + 2|\omega - \omega^{(0)}|$ in Step 6 is proposed by Ericsson and Ruhe [16].

(ii) The solution $\mathbf{x}_j$ of the symmetric linear system $G(\mathrm{i}\omega)\mathbf{x}_j = \mathbf{b}$ in Step 5.2 can be regarded as a deflated solution and can be solved by the $\mathrm{LDL}^\top$—factorization obtained by the $(k+1)$—step in Eq. (3.16) [17].

In summary, we have the following algorithm for solving the QEP (1.2).

Algorithm 3.2.

1. Compute $(\mathbf{X}_1, \mathbf{J}_1)$ by Algorithm 3.1, and choose a real symmetric positive semidefinite matrix $\mathbf{\Phi}$.
2. Compute the eigenvalues of the matrix pencil (3.18).
3. Compute $\Delta\mathbf{M}, \Delta\mathbf{G}, \Delta\mathbf{K}$ defined by Eqs. (3.14)–(3.16), and set

$$\mathbf{A} = \mathbf{M} + \Delta\mathbf{M} + \mathbf{G} + \Delta\mathbf{G} + \mathbf{K} + \Delta\mathbf{K}, \quad \mathbf{Q} = 2(\mathbf{M} + \Delta\mathbf{M} - \mathbf{K} - \Delta\mathbf{K}).$$

4. Apply Algorithm 2.1 to find the maximal solution $\mathbf{Z}_+$ for the NME (2.5).
5. Find the eigenvalues μ_j of $-\mathbf{Z}_+^{-1}\mathbf{A}$, for $j = 1, \ldots, n$.
6. Compute the eigenvalues $\lambda_i = (1 + \mu_i)/(1 - \mu_i)$ on the right half-plane, and obtain the eigenvalues on the left half-plane by symmetry.
7. Replace the eigenvalues of the matrix pencil (3.18) by those of $\mathbf{J}_1$.
8. If eigenvectors are required, compute the right and left eigenvectors $\mathbf{x}_i$ and $\mathbf{y}_i$ of $-\mathbf{Z}_+^{-1}\mathbf{A}$ corresponding to μ_j in Step 5 first, and then compute eigenvectors $\mathbf{x}_i$ and $(\lambda_i\mathbf{M} + \mathbf{M}\mathbf{S} + \mathbf{G})^{-\top}\bar{\mathbf{y}}_i$ of $G(\lambda)$ corresponding to $\pm\lambda_i$, respectively.
9. Replace the eigenvectors corresponding to the eigenvalues of the matrix pencil (3.18) by the columns of $\mathbf{X}_1$.

Remark. For the choice of $\mathbf{\Phi}$ in Step 1, numerical experiments show that letting $\mathbf{\Phi} = \gamma\mathbf{I}$ is generally good enough. We are to find a suitable γ such that the resulted eigenvalues of the pencil (3.18) after the Cayley

transformation are far from the unit circle which speeds up the convergence of Algorithm 2.1. If $\gamma = 0$, the eigenvalues of the pencil (3.18) are just those of J_1, which lie on the unit circle after the Cayley transformation. On the other hand, if γ is chosen large, the norms of $\Delta\mathbf{M}, \Delta\mathbf{G}, \Delta\mathbf{K}$ in Eqs. (3.14)–(3.16) may be too large to cause large errors to eigenvalues of the original pencil. Considering these two facts, we choose a γ between 1 and a suitable positive number so that the distance to the unit circle is maximized. Fortunately, numerical experiments show that a few steps of Fibonacci line search is sufficiently good to find a desired γ.

4. Numerical examples

To illustrate the performance of Algorithm 3.2, we present numerical results of three examples, using MATLAB 6.5 with machine accuracy $\varepsilon = 2.22 \times 10^{-16}$.

Example 1. Let

$$\mathbf{M}_1 = \mathbf{I}_6, \quad \mathbf{G}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \oplus \begin{bmatrix} -1 & 0 \\ 0 & b \end{bmatrix}$$

with $a = -5.000002000001$, $b = -16.000016000004$, $g = 9.000012000003$. We randomly generate an orthogonal matrix $\mathbf{Q}$ and define the coefficient matrices for the QEP (1.2) by

$$\mathbf{M} = \mathbf{Q}\mathbf{M}_1\mathbf{Q}^\top, \quad \mathbf{G} = \mathbf{Q}\mathbf{G}_1\mathbf{Q}^\top, \quad \mathbf{K} = \mathbf{Q}\mathbf{K}_1\mathbf{Q}^\top.$$

The QEP has four zero eigenvalues with partial multiplicities 2 and 2, and the other nonzero eigenvalues $\pm 1, \pm 1.000001, \pm 2, \pm 2.000001$.

Table 1 shows the numerical result by applying Algorithm 3.2 with $\mathbf{\Phi} = 50\mathbf{I}$ and Algorithm 3 in Ref. [1] (Guo's algorithm), respectively, to Example 1. Here and hereafter, 'No. of iterations' refers to the number of iteration steps of Algorithm 2.1 (SDA algorithm) applied to the NME (2.5), 'Residuals' refers to $\|(\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K})\mathbf{x}\|_2$ for an computed eigenpairs (λ, x), and 'Total flops' denotes the total flop counts for the computation of $(\mathbf{X}_1, \mathbf{J}_1)$ and the desired $\mathbf{Z}_+$ by Algorithm 3.1 and Algorithm 2.1, respectively. Algorithm 3.1 needs about $\frac{8}{3}n^3 + \mathrm{O}(n^2)$ flops to find $(\mathbf{X}_1, \mathbf{J}_1)$ for the zero eigenvalues. So, the total flops are $\frac{8}{3}n^3 + 6 \times \frac{7}{3}n^3 + \mathrm{O}(n^2)$.

Obviously, Algorithm 3.2 and Guo's algorithm converge with the same accuracy, while Algorithms 3.2 converges much faster than Guo's algorithm. The detailed convergence behaviors of both algorithms are illustrated in Table 2, where the quadratic convergence for Algorithm 3.2 and the linear convergence for Guo's algorithm are clearly shown.

Example 2. Let

$$\mathbf{M}_1 = \mathbf{I}_6, \quad \mathbf{G}_1 = \begin{bmatrix} 0 & \sqrt{15} \\ -\sqrt{15} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \sqrt{7} \\ -\sqrt{7} & 0 \end{bmatrix},$$

$$\mathbf{K}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -16 \end{bmatrix} \oplus \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \oplus \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Table 1
Numerical results of Example 1

	Algorithm 3.2	Guo's algorithm
No. of iterations	6	44
Residuals	$<10^{-13}$	$<10^{-13}$
Total flops	$\frac{50}{3}n^3 + \mathrm{O}(n^2)$	$\frac{308}{3}n^3 + \mathrm{O}(n^2)$

Table 2
Convergence behaviors of both algorithms for Example 1

Iteration steps	Algorithm 3.2	Iteration steps	Guo's algorithm
1	7.936×10^{-1}	⋮	⋮
2	6.343×10^{-3}	35	1.008×10^{-8}
3	2.186×10^{-4}	36	4.992×10^{-9}
4	1.185×10^{-7}	37	2.399×10^{-9}
5	1.043×10^{-14}	38	1.029×10^{-9}
6	0	⋮	⋮

Table 3
Numerical results of Example 2

	Algorithm 3.2	Guo's algorithm
No. of iterations	7	$>10\,000$
Residuals	$<10^{-12}$	>1
Total flops	$\frac{121}{3}n^3 + \mathrm{O}(n^2)$	****

Table 4
Convergence behaviors of Algorithm 3.2 for Example 2

Iteration steps	Algorithm 3.2
1	7.149×10^{-1}
2	6.465×10^{-2}
3	1.993×10^{-3}
4	2.018×10^{-6}
5	7.883×10^{-9}
6	1.203×10^{-13}
7	0

We randomly generate an orthogonal matrix $\mathbf{Q}$ and define

$$\mathbf{M} = \mathbf{Q}\mathbf{M}_1\mathbf{Q}^\top, \quad \mathbf{G} = \mathbf{Q}\mathbf{G}_1\mathbf{Q}^\top, \quad \mathbf{K} = \mathbf{Q}\mathbf{K}_1\mathbf{Q}^\top$$

for the coefficient matrices of the QEP (1.2). The QEP has two purely imaginary eigenvalues $\pm 2\mathrm{i}$ having partial multiplicities 1 and 1, and the other eigenvalues $\pm 2, \pm\frac{1}{2} \pm \frac{\sqrt{7}}{2}\mathrm{i}$ and $\pm\frac{\sqrt{7}}{2} \pm \frac{1}{2}\mathrm{i}$. Note that $\mathbf{K}$ is indefinite in this example, but $\mathbf{M} - \mathbf{K}$ is positive semidefinite. Hence both Algorithm 3.2 and Guo's algorithm still work for this example.

Inverse iteration of Step 4 in Algorithm 3.1 converges in 9 steps to $\pm 2\mathrm{i}$ and the associated eigenvectors. So, to find $(\mathbf{X}_1, \mathbf{J}_1)$ by Algorithm 3.1 needs about $\frac{72}{3}n^3 + \mathrm{O}(n^2)$ flops. Then we choose $\mathbf{\Phi} = \mathbf{I}$ in Algorithm 3.2 to compute the desired $\mathbf{Z}_+$ which needs about $7 \times \frac{7}{3}n^3 + \mathrm{O}(n^2)$ flops.

Tables 3 and 4 show the numerical results of Algorithm 3.2 and Guo's algorithm. Table 4 clearly shows the quadratic convergence of Algorithm 3.2, while Guo's algorithm does not converge after 10 000 steps, since two purely imaginary eigenvalues are transformed on the unit circle with partial multiplicities 1 and 1 by the Cayley transform, which violates the condition in Theorem 2.5.

Example 3. Consider the QEP (1.2) with coefficient matrices

$$\mathbf{M} = \mathbf{M}_1 \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{G} = \mathbf{G}_1 \oplus \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}, \quad \mathbf{K} = \mathbf{K}_1 \oplus \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix},$$

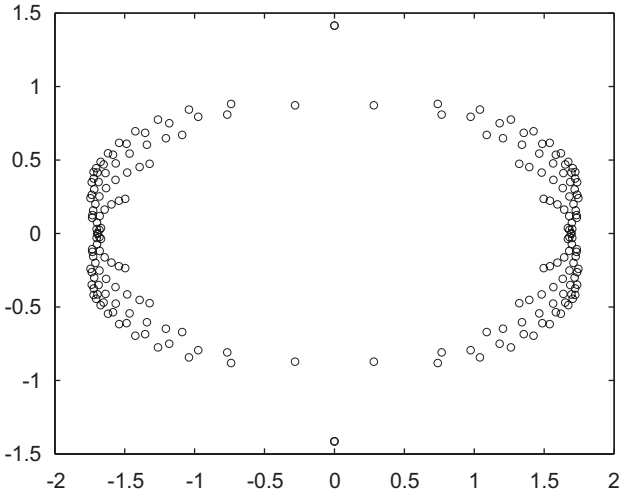

Fig. 1. Eigenvalues of Example 3.

Table 5
Numerical results of Example 3

	Algorithm 3.2	Guo's algorithm
No. of iterations steps	7	31
Residuals	$<10^{-15}$	$<10^{-15}$
Total flops	$\frac{89}{3}n^3 + \mathrm{O}(n^2)$	$\frac{217}{3}n^3 + \mathrm{O}(n^2)$

Table 6
Convergence behaviors of both algorithms for Example 3

Iteration steps	Algorithm 3.2	Iteration steps	Guo's algorithm
1	5.275×10^{-1}	⋮	⋮
2	9.049×10^{-2}	24	1.629×10^{-8}
3	8.476×10^{-3}	25	8.095×10^{-9}
4	1.562×10^{-4}	26	3.950×10^{-9}
5	1.964×10^{-6}	27	1.796×10^{-9}
6	7.544×10^{-11}	28	6.324×10^{-10}
7	0	⋮	⋮

where $\mathbf{M}_1, \mathbf{G}_1$ and $\mathbf{K}_1$ are 100×100 matrices chosen from Example 6.1 of Ref. [7]. The eigenvalues of the QEP computed by the Matlab function '**polyeig**' are plotted in Fig. 1, with two eigenvalues $\pm\sqrt{2}\mathrm{i}$ on the imaginary axis having partial multiplicities 2 and 2.

Inverse iteration of Step 4 in Algorithm 3.1 converges in 5 steps to obtain the purely imaginary eigenvalues $\pm\sqrt{2}\mathrm{i}$ and the associated eigenvectors. So, Algorithm 3.1 needs about $\frac{40}{3}n^3 + \mathrm{O}(n^2)$ flops for finding $(\mathbf{X}_1, \mathbf{J}_1)$. Next we choose $\mathbf{\Phi} = 50\mathbf{I}$ in Algorithm 3.2 to compute the desired $\mathbf{Z}_+$ which needs about $7 \times \frac{7}{3}n^3 + \mathrm{O}(n^2)$ flops.

Table 5 shows the numerical result of Algorithm 3.2 and Guo's algorithm, and Table 6 illustrates the convergence behaviors of both algorithms. Clearly, both algorithms converge with high accuracy, while Algorithm 3.2 converges much faster than Guo's algorithm. These three examples clearly show the benefit of shifting technique of purely imaginary eigenvalues.

5. Conclusions

In Ref. [1], Guo proposed an algorithm for computing all eigenvalues of a quadratic eigenvalue problem arising from gyroscopic systems, by finding a proper solvent of the quadratic matrix equation $\mathbf{MS}^2 + \mathbf{GS} + \mathbf{K} = \mathbf{0}$. Using a fine transformation, the problem can be transformed by solving the maximal solution of the nonlinear matrix equation $\mathbf{Z} + \mathbf{A}^\top \mathbf{Z}^{-1} \mathbf{A} = \mathbf{Q}$ with $\mathbf{A} = \mathbf{M} + \mathbf{G} + \mathbf{K}$ and $\mathbf{Q} = 2(\mathbf{M} - \mathbf{K})$. A cyclic reduction method, or the equivalent SDA method, can be applied. This approach preserves the Hamiltonian structure of the spectrum of the QEP, and is less expensive than the linearization approach followed by the QZ algorithm. However, it is based on the condition that the QEP has no eigenvalues on the imaginary axis. Although this approach still works for some cases when the condition is violated, the cyclic reduction or SDA method for the maximal solution of the nonlinear matrix equation converges much slower. In this paper, using the concept of model updating, we propose an eigenvalue shifting technique to transform the original gyroscopic system to a new gyroscopic system, shifting all purely imaginary eigenvalues to eigenvalues with nonzero real parts. Hence the SDA method can be applied to the new system with quadratic convergence rate. Numerical examples illustrate the efficiency of our approach.

Acknowledgments

We would like to thank Professor Eric K.W. Chu and the referees for their valuable comments and suggestions on the manuscript.

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